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Real Characters for Demazure Modules of Rank Two Affine Lie Algebras

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Using Littelmann's path model for highest weight representations of Kac–Moody algebras, we obtain explicit combinatorial expressions for certain specialized characters of all Demazure modules of $A_1^{(1)}$ and $A_2^{(2)}$.

1. INTRODUCTION

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra. Recall that for every dominant integral weight λ, there exists a unique (up to isomorphism) irreducible highest weight module $V = V(\lambda)$ of highest weight λ. In order to gain some insight into the structures of these modules, one can study their characters. The character of V, denoted $\chi(V)$, is the formal sum

$$\chi(V) = \sum_{\mu} (\dim V_\mu) e^{\mu}$$

over all weights μ, where V_μ is the weight space of V of weight μ and where e^{μ} is a formal exponential. This sum makes sense because each V_μ is finite-dimensional.

For symmetrizable Kac–Moody algebras, character formulas for all highest weight modules are known, the most familiar being Weyl's formula, due to Kac [Ka]. Unfortunately, the expressions for $\chi(V)$ are not combinatorial (they are alternating sums), thereby making it difficult to practically compute $\dim V_\mu$ for arbitrary weights μ.

An analogous problem exists for so-called Demazure modules, defined as follows. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ and let w be an element of

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the Weyl group W. The $\mathfrak{b}$-module generated by the one-dimensional extremal weight space $V_{w(\lambda)}$ is denoted by $E_w(\lambda)$ and called a *Demazure module*. They are finite-dimensional subspaces which form a filtration of V which is compatible with the Bruhat order of W, i.e., $E_w(\lambda) \subseteq E_{w'}(\lambda)$ whenever $w \leq w'$ for $w, w' \in W$, and $\bigcup_{w \in W} E_w(\lambda) = V$. Character formulas are also known for Demazure modules [D, Ku, Ma], but they are not combinatorial either.

This problem of finding combinatorial character formulas has already motivated a large body of work. Much of the research has not been focused on the (full) character, but on what are called specialized characters. A specialization of $\chi(V)$ is the image of $e^{-\lambda}\chi(V)$ under the map $e^{\mu} \mapsto q^{l(u)}$ where l is some integral linear function on the weight lattice. One then obtains a formal series in the single variable q which, hopefully, will be easier to study.

In this paper, we study the characters of all Demazure modules for the rank two affine Kac–Moody algebras, $A_1^{(1)}$ and $A_2^{(2)}$, and obtain surprisingly simple combinatorial expressions for a certain specialization of $\chi(E_w(\lambda))$, which we call the real character. For this specialization, which we believe has not been considered previously, we collect together all weight spaces whose weights differ only by an imaginary root. In other words, this specialization "ignores" the imaginary part of a weight, hence the name "real." Observe that this specialization may not be applied to the full module V because each V_n is infinite-dimensional; in fact, it is the only specialization with this property.

To formulate our main result we need some notation. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be the Cartan subalgebra. In the case of $A_2^{(2)}$, we choose the α_i such that $\langle \alpha_0, \alpha_1^{\vee} \rangle = -1$ and $\langle \alpha_1, \alpha_0^{\vee} \rangle = -4$, where the $\alpha_i^{\vee}$ are the simple coroots. Let $\delta \in \mathfrak{h}^*$ denote the imaginary root $\delta = \alpha_0 + \alpha_1$ if $\mathfrak{g} = A_1^{(1)}$ and $\delta = \alpha_0 + 2\alpha_1$ if $\mathfrak{g} = A_2^{(2)}$. Set

$$E_w(\lambda)_n := \bigoplus_{\mu \in \lambda - n\alpha_0 + \mathbb{Z}\delta} E_w(\lambda)_\mu.$$

Then, the real character of a Demazure module is

$$\chi_r(E_w(\lambda)) = \sum_{n \in \mathbb{Z}} \dim E_w(\lambda)_n q^n.$$

Let $\Lambda_0, \Lambda_1 \in \mathfrak{h}^*$ represent the two fundamental weights, defined by $\langle \Lambda_i, \alpha_j^{\vee} \rangle = \delta_{ij}$. Let r_i denote the simple reflection with respect to α_i. For every $n > 0$, the Weyl group W has two elements of length n,

$$w_n^+ := \underbrace{\cdots r_0 r_1 r_0}_{n} \qquad \text{and} \qquad w_n^- := \underbrace{\cdots r_1 r_0 r_1}_{n}.$$

For $m \geq n \geq 0$, and for q an indeterminate, let

$$\begin{bmatrix} m \\ n \end{bmatrix} := \frac{(q^m - 1)(q^{m-1} - 1) \cdots (q^{m-n+1} - 1)}{(q-1)(q^2-1) \cdots (q^n - 1)}$$

be the q-binomial coefficients. For ease of notation, we set $[m] := \left[\begin{smallmatrix} m \\ 1 \end{smallmatrix}\right]$. Also, $[m]_{q^2}$ will denote the expression $\left[\begin{smallmatrix} m \\ 1 \end{smallmatrix}\right]$ with q replaced by q^2.

THEOREM 1. *Let* $\lambda = s\Lambda_0 + t\Lambda_1$. *Then*

(1.1) *If* $\mathfrak{g} = A_1^{(1)}$, *then*

$$\chi_r(E_w(\lambda)) = \begin{cases} q^{-t-(s+t)k}[t+1][s+t+1]^{n-1} \\ \qquad \text{if } w = w_n^- \text{ where } n = 2k+1 \text{ or } 2k+2; \\ q^{-(s+t)k}[s+1][s+t+1]^{n-1} \\ \qquad \text{if } w = w_n^+ \text{ where } n = 2k > 0 \text{ or } 2k+1. \end{cases}$$

(1.2) *If* $\mathfrak{g} = A_2^{(2)}$, *then*

$$\chi_r(E_w(\lambda)) = \begin{cases} q^{-(s+2t)k}[s+1]\begin{bmatrix} 2t+s+2 \\ 2 \end{bmatrix}^k \\ \qquad \text{if } w = w_{2k+1}^+ \text{ for some } k \geq 0; \\ q^{-(s+2t)k}\dfrac{[s+1][2t+s+2]}{(q+1)}\begin{bmatrix} 2t+s+2 \\ 2 \end{bmatrix}^{k-1} \\ \qquad \text{if } w = w_{2k}^+ \text{ for some } k \geq 1; \\ q^{-2t-(s+2t)k}[t+1]_{q^2}\begin{bmatrix} 2t+s+2 \\ 2 \end{bmatrix}^k \\ \qquad \text{if } w = w_{2k+1}^- \text{ for some } k \geq 0; \\ q^{-2t-(s+2t)(k-1)}[t+1]_{q^2}[2t+s+1]\begin{bmatrix} 2t+s+2 \\ 2 \end{bmatrix}^{k-1} \\ \qquad \text{if } w = w_{2k}^- \text{ for some } k \geq 1. \end{cases}$$

These expressions are indeed combinatorial. In fact, in (1.2), we always have that either $s+1$ or $2t+s+2$ is even and for n even, we have $[n/2]_{q^2} = [n]/(q+1)$.

The main tool that we use to obtain these characters is Littelmann's path model for highest weight representations [L1, L2]. The path model is a recently discovered combinatorial parametrization for bases of integrable modules by certain piecewise linear paths whose images line in $\mathfrak{h}^*$. A brief summary of this theory is given in the Section 2. The path model allows us to calculate $\chi_r(E_w(\lambda))$ recursively over w and λ.

In a previous paper [S] we already calculated the dimension of these Demazure modules. Since the value of a real character at $q = 1$ is just the dimension, the present paper generalizes the results of [S]. The method of the proofs are similar except that, for the computation of the real character, some new recursion identities (see br3–br6) are required which have no analogue in [S].

2. THE PATH MODEL

The path model, introduced in [L1, L2], gives a combinatorial parametrization of the basis vectors of a highest weight module for a Kac–Moody algebra. These basis vectors are parametrized by certain piecewise linear paths $\pi: [0,1] \to \mathfrak{h}^*$. We now give a brief synopsis of those parts of the theory necessary for this paper. (See [L1, L2] for more details.)

2.1. *General Theory*

Let Π denote the set of all continuous piecewise linear paths π: $[0,1] \to \mathfrak{h}^*$ with $\pi(0) = 0$ and $\pi(1)$ a weight. We identify any two such paths if their images coincide. For any two paths $\tau_1, \tau_2 \in \Pi$, we denote by $\tau_1 * \tau_2$ their concatenation,

$$(\tau_1 * \tau_2)(t) := \begin{cases} \tau_1(2t) & \text{for } t \in [0, 1/2]; \\ \tau_1(1) + \tau_2(2t-1) & \text{for } t \in [1/2, 1]. \end{cases}$$

For any simple root α_i, let $r_i \in W$ denote the reflection with respect to α_i. For any path $\tau \in \Pi$ and any α_i, we define the path $r_i\tau$ by $(r_i\tau)(t) := r_i(\tau(t))$. Let $m_i(\tau) := \min_{t \in [0,1]} \langle \tau(t), \alpha_i^\vee \rangle$.

Now let τ be a path. Let α_i be a simple root. We will now describe a new path $f_i\tau$. Let $p \in [0,1]$ be maximal such that $\langle \tau(p), \alpha_i^\vee \rangle = m_i(\tau)$. If $\langle \tau(1), \alpha_i^\vee \rangle - m_i(\tau) < 1$, then $f_i\tau$ is undefined. If not, we do the following. Let $x \in [p, 1]$ be minimal such that $\langle \tau(x), \alpha_i^\vee \rangle = m_i(\tau) + 1$. We now "cut" τ into three parts τ_1, τ_2, and τ_3. Each of these parts is a path in Π and they are defined by:

$$\tau_1(t) := \tau(tp); \qquad \tau_2(t) := \tau(p + t(x-p)) - \tau(p);$$
$$\tau_3(t) := \tau(x + t(1-x)) - \tau(x).$$

Notice that $\tau = \tau_1 * \tau_2 * \tau_3$. We define $f_i\tau := \tau_1 * r_i\tau_2 * \tau_3$. Note that $f_i\tau(1) = \tau(1) - \alpha_i$. The f_i: $\Pi \to \Pi$ are called root operators.

Choose a path $\pi \in \Pi$ whose image lies in the fundamental chamber and whose endpoint $\pi(1)$ is a dominant weight λ. Such a path, π, is called a dominant path. Let $B\pi = \{f_{i_1} \cdots f_{i_k}\pi | i_j \in [0, \dots, n-1], k \geq 0\}$ be the set of all possible paths that one obtains by applying the root operators to π. We call $B\pi$ the path model associated to π. We then have the following theorem of Littelmann [L1, Thm. 5.6]:

THEOREM 2. $\chi(V) = \sum_{\tau \in B\pi} e^{\tau(1)}$.

We also have an analogue for Demazure modules. Let $w = r_{j_1} \cdots r_{j_k}$ be a reduced decomposition of w. Let $P_w\pi = \{f_{j_1}^{i_1} \cdots f_{j_k}^{i_k}\pi | i_1, \dots, i_k \in \mathbb{Z}_{\geq 0}\}$. Then, by Theorem 5.2 of Littelmann [L1],

THEOREM 3. $\chi(E_w(\lambda)) = \sum_{\tau \in P_w\tau} e^{\pi(1)}$.

From this description, we see that, once we have chosen a dominant path π, the problem is to characterize all of the paths contained in $B\pi$ (or $P_w\pi$). In general, this is not an easy task. However, for one particular type of dominant path, this has already been done for us [L1]. The dominant path in this case is the straight path $\pi_\lambda(t) = t\lambda$ from 0 to λ. The paths in the associated path model are called Lakshmibai–Seshadri paths.

2.2. *Lakshmibai–Seshadri Paths*

For a given $\lambda \in P^+$, we choose as dominant path $\pi_\lambda(t) = t\lambda$. The elements in $B\pi_\lambda$ are called Lakshmibai–Seshadri (or L-S) paths of shape λ. These paths can be characterized as follows. There is a bijection between paths $\tau \in B\pi_\lambda$ and pairs of sequences $(\boldsymbol{\sigma}, \mathbf{a})$ that satisfy the conditions

$$\boldsymbol{\sigma}: \sigma_1 > \cdots > \sigma_n, \qquad \sigma_i \in W/W_\lambda \ \forall i$$
$$\mathbf{a}: a_0 = 0 < a_1 < \cdots < a_n = 1, \qquad a_i \in \mathbb{Q} \ \forall i,$$

where W_λ is the stabilizer of λ in W and where $>$ is the relative Bruhat order. In addition, we require that for every $i \in [1, \dots, n-1]$, there exists an a_i-chain for the pair (σ_i, σ_{i+1}). Recall that, by definition [L1], the existence of an a-chain for a pair (σ, σ') of cosets in W/W_λ means that there exists a sequence of cosets in W/W_λ

$$\kappa_0 := \sigma > \kappa_1 := r_{\beta_1}\sigma > \kappa_2 := r_{\beta_2}r_{\beta_1}\sigma > \kappa_s := r_{\beta_s} \cdots r_{\beta_1}\sigma = \sigma',$$

where the r_{β_i} are the reflections with respect to the positive real roots β_i, such that for all $i \in [1, \dots, s]$

$$l(\kappa_i) = l(\kappa_{i-1}) - 1 \qquad \text{and} \qquad a\langle \kappa_i(\lambda), \beta_i^\vee \rangle \in \mathbb{Z}.$$

We shall also refer to such a pair $(\boldsymbol{\sigma}, \mathbf{a})$ as Lakshmibai–Seshadri (or L-S). For any such pair, the corresponding path is

$$\sigma(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1}) \sigma_i(\lambda) + (t - a_{j-1}) \sigma_j(\lambda) \qquad \text{for } a_{j-1} \le t \le a_j.$$

By Theorem 5.2 of Littelmann [L1], we have that $P_w \pi_\lambda = \{(\boldsymbol{\sigma}, \mathbf{a}) \text{ L-S} \mid \sigma_1 \le w\}$.

3. DESCRIPTION OF L-S PATHS FOR THE BASIC MODULES

From here on, we assume that $\mathfrak{g}$ is isomorphic to $A_1^{(1)}$ or to $A_2^{(2)}$. In this section, we will work directly from the definition of L-S pairs in order to obtain an explicit characterization of all such pairs of shape Λ_0 or of shape Λ_1. Note that r_0 fixes Λ_1 and r_1 fixes Λ_0. We choose the following coset representatives for W/W_{Λ_0} and W/W_{Λ_1}:

$$W/W_{\Lambda_0} = \left\{ w_n^+ := r_{i_n} \cdots r_{i_2} r_{i_1} \mid n \in \mathbb{N} \text{ and } i_j = j + 1 \bmod 2 \right\};$$

$$W/W_{\Lambda_1} = \left\{ w_n^- := r_{i_n} \cdots r_{i_2} r_{i_1} \mid n \in \mathbb{N} \text{ and } i_j = j \bmod 2 \right\}.$$

On each of these two coset spaces, the Bruhat order is a total ordering. For $\epsilon \in \{-, +\}$, $w_n^\epsilon > w_m^\epsilon \Leftrightarrow n > m$. When $\mathfrak{g} = A_1^{(1)}$, we have $\langle \alpha_0, \alpha_1^\vee \rangle = \langle \alpha_1, \alpha_0^\vee \rangle = -2$. In the case of $A_2^{(2)}$, to fix notation, we choose the α_i such that $\langle \alpha_0, \alpha_1^\vee \rangle = -1$ and $\langle \alpha_1, \alpha_0^\vee \rangle = -4$. For $n \ge 0$, set:

Formulas 1.

$$d_{2n}^+ := \left\langle w_{2n}^+(\Lambda_0), \alpha_1^\vee \right\rangle = \langle \alpha_0, \alpha_1^\vee \rangle n$$

$$d_{2n+1}^+ := \left\langle w_{2n+1}^+(\Lambda_0), \alpha_0^\vee \right\rangle = -(2n+1)$$

$$d_{2n}^- := \left\langle w_{2n}^-(\Lambda_1), \alpha_0^\vee \right\rangle = \langle \alpha_1, \alpha_0^\vee \rangle n$$

$$d_{2n+1}^- := \left\langle w_{2n+1}^-(\Lambda_1), \alpha_1^\vee \right\rangle = -(2n+1).$$

LEMMA 1. *Let Λ be a fundamental weight. The L-S paths π of shape Λ are those paths $\pi = (\boldsymbol{\sigma}, \mathbf{a})$ such that*

$$\boldsymbol{\sigma}: w_{n+k}^{\epsilon} > w_{n+k-1}^{\epsilon} > w_{n+k-2}^{\epsilon} > \cdots > w_{n}^{\epsilon}, \qquad n, k \geq 0$$
$$\mathbf{a}: 0 < a_{n+k}^{\epsilon} < a_{n+k-1}^{\epsilon} < \cdots < a_{n+1}^{\epsilon} < 1,$$

where $\epsilon = +$ if $\Lambda = \Lambda_0$, $\epsilon = -$ if $\Lambda = \Lambda_1$, and where $a_j^{\epsilon} \cdot d_j^{\epsilon} \in \mathbb{Z}$.

See [S] for proof.

For any L-S path $\pi = (\boldsymbol{\sigma}, \mathbf{a})$ where $\boldsymbol{\sigma}: \sigma_1 > \cdots > \sigma_r$ define $\operatorname{beg}(\pi) := \sigma_1$ and $\operatorname{end}(\pi) := \sigma_r$. For Λ one of the fundamental weights and for $m \geq n \geq 0$, let

$$p_{\Lambda}(m, n) := \{\pi = (\boldsymbol{\sigma}, \mathbf{a}) \text{ L-S of shape } \Lambda | \\ \operatorname{beg}(\pi) = w_m^{\epsilon} \text{ and} \operatorname{end}(\pi) = w_n^{\epsilon}\}.$$

We will write $P_n(\pi_{s\Lambda_0})$ for $P_{w_n^+}(\pi_{s\Lambda_0})$ and $P_n(\pi_{s\Lambda_1})$ for $P_{w_n^-}(\pi_{s\Lambda_1})$. Also, for any set S of paths in the path model $B\pi_{\Lambda}$, let

$$\chi(S) = e^{-\Lambda} \sum_{\tau \in S} e^{\tau(1)}.$$

We will refer to $\chi(S)$ as the character of S. To simplify notation, we will set $e^{-\alpha_0} := u$ and $e^{-\alpha_1} := v$. We also set $c_{\Lambda}(m, n) := \chi(p_{\Lambda}(m, n))$.

In order to determine the characters of any given Demazure module of $\mathfrak{g}$, we will first calculate $c_{\Lambda_0}(m, n)$ and $c_{\Lambda_1}(m, n)$ for $m \geq n \geq 0$. Notice that $c_{\Lambda_0}(m, n)$ and $c_{\Lambda_1}(m, n)$ are elements of $\mathbb{N}[u, v]$. We do this for $A_1^{(1)}$ and $A_2^{(2)}$ in Sections 4–5. Notice that these results will give us information about the characters of any Demazure module for Λ_0 and Λ_1 because

$$\chi\left(E_{w_n^+}(\Lambda_0)\right) = \chi\left(P_n(\pi_{\Lambda_0})\right) = \sum_{0 \leq j \leq i \leq n} \chi\left(p_{\Lambda_0}(i, j)\right) = \sum_{0 \leq j \leq i \leq n} c_{\Lambda_0}(i, j)$$

and similarly for $\chi(E_{w_n^-}(\Lambda_1))$.

4. CHARACTERS FOR THE BASIC MODULES OF $A_1^{(1)}$

In this section, we establish recurrence relations for the $c_{\Lambda}(m, n)$. These relations will be used to prove Theorem 1.

PROPOSITION 1. *Let $\mathfrak{g} = A_1^{(1)}$. The $c_{\Lambda_0}(m, n)$ satisfy the following recursion relations*:

R1.1. $c_{\Lambda_0}(m, 2j) = u^{2j}v^{2j}(c_{\Lambda_0}(m-2, 2j) + c_{\Lambda_0}(m-2, 2j-1)) + u^{2j-1}v^{2j}(c_{\Lambda_0}(m-2, 2j-1) + c_{\Lambda_0}(m-2, 2j-2))$;

R1.2. $c_{\Lambda_0}(m, 2j+1) = u^{2j+1}v^{2j+1}(c_{\Lambda_0}(m-2, 2j+1) + c_{\Lambda_0}(m-2, 2j)) + u^{2j+1}v^{2j}(c_{\Lambda_0}(m-2, 2j) + c_{\Lambda_0}(m-2, 2j-1))$.

Proof. Consider the set $p_{\Lambda_0}(m,n)$. By definition, a path $\pi = (\boldsymbol{\sigma}, \mathbf{a})$ is an element of $p_{\Lambda_0}(m,n)$ if and only if

$$\boldsymbol{\sigma}: w_n^+ > w_{m-1}^+ > w_{m-2}^+ > \cdots > w_n^+, \qquad m \geq n \geq 0$$

$$\mathbf{a}: 0 < \frac{i_m}{m} < \frac{i_{m-1}}{m-1} < \cdots < \frac{i_{n+1}}{n+1} < 1,$$

where $i_j \in \mathbb{N}$ for all j. A $(m-n)$-tuple $(i_m, i_{m-1}, \dots, i_{n+1}) \in \mathbb{N}^{m-n}$ satisfies these inequalities if and only if $1 \leq i_m \leq i_{m-1} \leq \cdots \leq i_{n+1} \leq n$. Let $S = S(m,n)$ denote the set of all such $(m-n)$-tuples. For any $(i_m, i_{m-1}, \dots, i_{n+1}) \in S$, the weight of the associated path π is

$$\pi(1) = w_n^+(\Lambda_0) - \sum_{\substack{j \text{ even} \\ m \geq j \geq n+1}} i_j \alpha_1 - \sum_{\substack{j \text{ odd} \\ m \geq j \geq n+1}} i_j \alpha_0 .$$

Now write S as the disjoint union of four subsets S_1, S_2, S_3, and S_4 where

$$S_1 := \{1 \leq i_m \leq \cdots \leq i_{n+3} \leq n, i_{n+2} = i_{n+1} = n\};$$

$$S_2 := \{1 \leq i_m \leq \cdots \leq i_{n+2} \leq n-1, i_{n+1} = n\};$$

$$S_3 := \{1 \leq i_m \leq \cdots \leq i_{n+2} \leq n-1, i_{n+1} = n-1\};$$

$$S_4 := \{1 \leq i_m \leq \cdots \leq i_{n+1} \leq n-2\}.$$

We first consider the set S_1. The map $(i_m, \dots, i_{n+3}, n, n) \mapsto (i_m, \dots, i_{n+3})$ is a bijection of S_1 onto $S(m-2,n)$. For any -tuple $(i_m, \dots, i_{n+3}, n, n) \in S_1$, the corresponding path $\pi_1(1)$ in $p(m,n)$ has weight

$$\pi_1(1) = w_n^+(\Lambda_0) - \left(\sum_{\substack{j \text{ even} \\ m \geq j \geq n+3}} i_j \right) \alpha_1 - \left(\sum_{\substack{j \text{ odd} \\ m \geq j \geq n+3}} i_j \right) \alpha_0 - n\alpha_0 - n\alpha_1 .$$

For any -tuple $(i_m, \dots, i_{n+3}) \in S(m-2,n)$, the corresponding path $\pi_2(1)$ in $p(m-2,n)$ has weight

$$\pi_2(1) = w_n^+(\Lambda_0) - \left(\sum_{\substack{j \text{ even} \\ m \geq j \geq n+3}} i_j \right) \alpha_1 - \left(\sum_{\substack{j \text{ odd} \\ m \geq j \geq n+3}} i_j \right) \alpha_0 .$$

The difference between the weights of these two paths is $\pi_1(1) - \pi_2(1) = -n\alpha_0 - n\alpha_1$. Therefore, $\chi(S_1) = u^n v^n c(m-2,n)$. We use the same type of reasoning to determine $\chi(S_i)$ for the other i. The map $(i_m, \dots, i_{n+2}, n)$ $\mapsto (i_m, \dots, i_{n+2})$ is a bijection of S_2 onto $S(m-2, n-1)$. From this we obtain $\chi(S_2) = u^n v^n c(m-2, n-1)$. The map $(i_m, \dots, i_{n+2}, n-1) \mapsto$

$(i_m, \ldots, i_{n+2})$ is a bijection of S_3 onto $S(m-2, n-1)$ giving us $\chi(S_3) = u^n \ v^{n-1} c(m-2, n-1)$ if n is odd and $\chi(S_3) = u^{n-1} v^n c(m-2, n-1)$ if n is even. The set S_4 equals $S(m-2, n-2)$. We have $\chi(S_4) = u^n v^{n-1} c(m-2, n-2)$ if n is odd and $\chi(S_4) = u^{n-1} v^n c(m-2, n-2)$ if n is even. Adding the $\chi(S_i)$ gives us the desired result. ∎

The symmetric roles of α_0 and α_1 give us the following corollary.

PROPOSITION 2. *Let $\mathfrak{g} = A_1^{(1)}$. The $c_{\Lambda_1}(m, n)$ satisfy the following recursion relations*:

R1.3. $c_{\Lambda_1}(m, 2j) = u^{2j} \ v^{2j}(c_{\Lambda_1}(m-2, 2j) + c_{\Lambda_1}(m-2, 2j-1)) + u^{2j} v^{2j-1}(c_{\Lambda_1}(m-2, 2j-1) + c_{\Lambda_1}(m-2, 2j-2))$;

R1.4. $c_{\Lambda_1}(m, 2j+1) = u^{2j+1} v^{2j+1}(c_{\Lambda_1}(m-2, 2j+1) + c_{\Lambda_1}(m-2, 2j)) + u^{2j} v^{2j+1}(c_{\Lambda_1}(m-2, 2j) + c_{\Lambda_1}(m-2, 2j-1))$.

5. CHARACTERS FOR THE BASIC MODULES OF $A_2^{(2)}$

In Section 5.1, we establish recurrence relations for the $c_{\Lambda_0}(m, n)$. In Section 5.2, we establish recurrence relations for a certain specialization for the $c_{\Lambda_1}(m, n)$. These relations will be used to prove Theorem 1.

5.1. *Basic Modules of Highest Weight Λ_0*

PROPOSITION 3. *Let $\mathfrak{g} = A_2^{(2)}$. The $c_{\Lambda_0}(m, n)$ satisfy the following recursion relations*:

R2.1. $c_{\Lambda_0}(m, 2k) = u^{2k} v^k (c_{\Lambda_0}(m-2, 2k) + c_{\Lambda_0}(m-2, 2k-1)) + u^{2k-1} v^k (c_{\Lambda_0}(m-2, 2k-1) + c_{\Lambda_0}(m-2, 2k-2))$;

R2.2. $c_{\Lambda_0}(m, 2k+1) = u^{2k+1} v^k (c_{\Lambda_0}(m-2, 2k) + c_{\Lambda_0}(m-2, 2k-1))$.

Proof. Consider the set $p_{\Lambda_0}(m, n)$. By definition, a path $\pi = (\boldsymbol{\sigma}, \mathbf{a})$ is an element of $p_{\Lambda_0}(m, n)$ if and only if

$$\boldsymbol{\sigma}: w_m^+ > w_{m-1}^+ > w_{m-2}^+ > \cdots > w_n^+, \qquad m \ge n \ge 0$$

$$\mathbf{a}: 0 < \frac{i_m}{m} < \frac{i_{m-1}}{m-1} < \cdots < \frac{i_{n+1}}{n+1} < 1,$$

where i_j is an integer and is even whenever j is even. Then $(i_m, \ldots, i_{n+1}) \in \mathbb{N}^{m-n}$ satisfies these inequalities if and only if $1 \le i_m \le i_{m-1} \le \cdots \le i_{n+1} \le n$ where i_j is even if j is even. For any given m and n, let

$S = S(m,n)$ denote the set of all such $(m-n)$-tuples. Notice that, for any given $(i_m, \ldots, i_{n+1}) \in S$, the weight of an associated path π is

$$\pi(1) = w_n^+(\Lambda_0) - \frac{1}{2} \sum_{\substack{j \text{ even} \\ m \geq j \geq n+1}} i_j \alpha_1 - \sum_{\substack{j \text{ odd} \\ m \geq j \geq n+1}} i_j \alpha_0 .$$

The rest of the proof uses the same methods as in Proposition 1. When n is odd, we write S as the disjoint union of two subsets S_1 and S_2 where

$$S_1 := \{1 \leq i_m \leq \cdots \leq i_{n+2} \leq n-1, i_{n+1} = n-1\};$$
$$S_2 := \{1 \leq i_m \leq \cdots \leq i_{n+1} \leq n-3\}.$$

When n is even we write S as the disjoint union of four subsets S_1, S_2, S_3, and S_4 where

$$S_1 := \{1 \leq i_m \leq \cdots \leq i_{n+3} \leq n, i_{n+2} = i_{n+1} = n\};$$
$$S_2 := \{1 \leq i_m \leq \cdots \leq i_{n+2} \leq n-2, i_{n+1} = n\};$$
$$S_3 := \{1 \leq i_m \leq \cdots \leq i_{n+2} \leq n-2, i_{n+1} = n-1\};$$
$$S_4 := \{1 \leq i_m \leq \cdots \leq i_{n+1} \leq n-2\}. \qquad \blacksquare$$

5.2. *Basic Modules of Highest Weight* Λ_1

The $c_{\Lambda_1}(m,n)$ do not seem to satisfy simple recursion relations as do the $c_{\Lambda_0}(m,n)$. However, certain specializations of the $c_{\Lambda_1}(m,n)$ satisfy recursion relations and it is these, in the end, that we will prove (in Proposition 4).

By Lemma 1, if $\pi = (\boldsymbol{\sigma}, \mathbf{a}) \in p_{\Lambda_1}(m,n)$, then $\mathbf{a}$ satisfies

$$0 < a_m < a_{m-1} < \cdots < a_{n+1} < 1,$$

where $2ja_j \in \mathbb{Z}$ if j is even and $ja_j \in \mathbb{Z}$ if j is odd. Setting $i_j = 2ja_j$, we obtain an equivalent set of inequalities

$$0 < \frac{i_m}{m} < \frac{i_{m-1}}{m-1} < \cdots < \frac{i_{n+1}}{n+1} < 2,$$

where i_j is even if j is odd.

For m and n fixed, let $S = S(m,n)$ be the set of $(m-n)$-tuples $(i_m, \ldots, i_{n+1}) \in \mathbb{N}^{m-n}$ that satisfy the inequalities $\mathbf{a}$ and the congruence conditions. We will also identify S with the set of paths in $p_{\Lambda_1}(m,n)$ that it determines. So, the expression $\chi(S)$ makes sense. For each $j \in \{n, \ldots, m\}$ look at the subset $S_j \subseteq S$ of elements that satisfy

$$0 < \frac{i_m}{m} < \cdots < \frac{i_{j+1}}{j+1} \le 1 < \frac{i_j}{j} < \cdots < \frac{i_{n+1}}{n+1} < 2.$$

Note that $S = \dot{\bigcup}_{n \le j \le m} S_j$. Now, $(i_m, \ldots, i_{n+1}) \in S_j$ if and only if

$$\begin{aligned} 1 \le i_m \le i_{m-1} \le \cdots \le i_{j+1} \le j + 1 \le i_j \le i_{j-1} + 1 \le \cdots \\ \le i_{n+1} + j - n - 1 \le n + j. \end{aligned}$$

The weight of the associated path is

$$\begin{aligned} \pi(1) = \frac{1}{2}\Bigg(&\frac{i_m}{m} w_m^-(\Lambda_1) + \left(\frac{i_{m-1}}{m-1} - \frac{i_m}{m}\right) w_{m-1}^-(\Lambda_1) + \cdots + \left(1 - \frac{i_{j+1}}{j+1}\right) \\ &\times w_j^-(\Lambda_1) + \left(\frac{i_j}{j} - 1\right) w_j^-(\Lambda_1) + \cdots + \left(2 - \frac{i_{n+1}}{n+1}\right) w_n^-(\Lambda_1)\Bigg). \end{aligned}$$

Fix j. In order to determine $\chi(S_j)$, we will "cut" the path in two, where for any $\{i_m, \ldots, i_{n+1}\} \in S_j$, the weight of the first half of the path is

$$\begin{aligned} &\frac{1}{2}\left(\frac{i_m}{m} w_m^-(\Lambda_1) + \left(\frac{i_{m-1}}{m-1} - \frac{i_m}{m}\right) w_{m-1}^-(\Lambda_1) + \cdots + \left(1 - \frac{i_{j+1}}{j+1}\right) w_j^-(\Lambda_1)\right) \\ &\quad = \frac{1}{2} w_j^-(\Lambda_1) - \sum_{\substack{k \text{ even} \\ m \ge k \ge j+1}} i_k \alpha_0 - \frac{1}{2} \sum_{\substack{k \text{ odd} \\ m \ge k \ge j+1}} i_k \alpha_1. \end{aligned}$$

Let $a(m, j) = \sum_\mu e^\mu$ where we sum over all such weights. The weight of the second half of the path is

$$\begin{aligned} &\frac{1}{2}\left(\left(\frac{i_j}{j} - 1\right) w_j^-(\Lambda_1) + \cdots + \left(2 - \frac{i_{n+1}}{n+1}\right) w_n^-(\Lambda_1)\right) \\ &\quad = w_n^-(\Lambda_1) - \sum_{\substack{k \text{ even} \\ j \ge k \ge n+1}} i_k \alpha_0 - \frac{1}{2} \sum_{\substack{k \text{ odd} \\ j \ge k \ge n+1}} i_k \alpha_1 - \frac{1}{2} w_j^-(\Lambda_1). \end{aligned}$$

Likewise, let $b(j, n) = \sum_\mu e^\mu$ where we sum over all such weights. Because $\chi(S_j) = a(m, j) \cdot b(j, n)$, it suffices to study $a(m, j)$ and $b(j, n)$. Using the same methods as in Proposition 1, we obtain the following two lemmas.

LEMMA 2. *The $a(m,j)$ satisfy the following recursion relations*:

a1. $a(m,2l) = u^{2l}v^{l}a(m-2,2l-1) + u^{2l}v^{(2l-1)/2}a(m-2,2l-2))$;

a2. $a(m,2l+1) = u^{2l+2}v^{l+1}a(m-2,2l+1) + u^{2l+2}v^{(2l+1)/2}a(m-2,2l) + u^{2l+1}v^{(2l+1)/2}a(m-2,2l) + u^{2l}v^{(2l+1)/2}a(m-2,2l-1)$.

LEMMA 3. *The $b(j,n)$ satisfy the following recursion relations*:

b1. $b(j,2l) = u^{2l}v^{(2l-1)/2}(b(j-2,2l-1) + b(j-2,2l-2))$;

b2. $b(j,2l+1) = u^{2l+1}v^{(2l+1)/2}\ (b(j-2,2l+1) + b(j-2,2l)) + u^{2l}v^{(2l+1)/2}(b(j-2,2l) + b(j-2,2l-1))$.

Let $a_r(m,j)$ and $b_r(j,n)$ denote the specialized characters of $a(m,n)$ and $b(m,n)$ obtained by setting $e^{-\alpha_0} := q$ and $e^{-\alpha_1} := q^{-2}$. A consequence of Lemmas 2 and 3 is

LEMMA 4. *The $a_r(m,j)$ and $b_r(j,n)$ satisfy the following recursion relations*:

ar1. $a_r(m,2k) = a_r(m-2,2k-1) + qa_r(m-2,2k-2))$;

ar2. $a_r(m,2k+1) = a_r(m-2,2k+1) + (1+q)a_r(m-2,2k) + q^{-1}a_r(m-2,2k-1)$;

br1. $b_r(j,2k) = q(b_r(j-2,2k-1) + b_r(j-2,2k-2))$;

br2. $b_r(j,2k+1) = b_r(j-2,2k+1) + (1+q^{-1})b_r(j-2,2k) + q^{-1}b_r(j-2,2k-1)$.

We will also need the following relations:

LEMMA 5. *The $b_r(j,n)$ satisfy*

br3. $b_r(2l+2,2k) + b_r(2l+1,2k-2) = q^2(b_r(2l+1,2k-1) + b_r(2l+1,2k-2))$;

br4. $(1+q)b_r(2l+1,2k) = b_r(2l,2k) + b_r(2l,2k-1)$;

br5. $b_r(2l,2k-1) + (1+q)b_r(2l+1,2k+1) = (1+q^{-1})b_r(2l,2k+1) + (q^{-2}+q^{-1})b_r(2l,2k) + q^{-2}b_r(2l,2k-1)$;

br6. $b_r(2l+2,2k+1) = qb_r(2l+1,2k+1) + (1+q)b_r(2l+1,2k)$.

Proof. The proof for each of these relations is by induction on $j+n$. It is easy to see that these relations hold for small values of j and n. The proof for each of the br3–br6 uses the same method. We will just show br3. We suppose that, for $j + n$ up to a certain value, br3–br6 hold:

$$\begin{aligned} b_r(2l+2, 2k) &+ b_r(2l+1, 2k-2) \\ &= q\big(b_r(2l, 2k-1) + b_r(2l, 2k-2) + b_r(2l-1, 2k-3) \\ &\qquad + b_r(2l-1, 2k-4)\big) \qquad (1) \\ &= q\big(qb_r(2l-1, 2k-1) + (1+q)b_r(2l-1, 2k-2) \\ &\qquad + b_r(2l-1, 2k-3)\big) \\ &\quad + q^3 b_r(2l-1, 2k-3) + q^3 b_r(2l-1, 2k-4) \qquad (2) \\ &= q^2 b_r(2l+1, 2k-1) + q^2 b_r(2l+1, 2k-2). \qquad (3) \end{aligned}$$

We use br1 to obtain (1), relations br3 and br6 (which are true by induction) to obtain (2), relations br1 and br2 to obtain (3). ∎

Let $cr_{\Lambda_1}(m,n)$ be the specialized character of $p_{\Lambda_1}(m,n)$ obtained by setting $e^{-\alpha_0} := q$ and $e^{-\alpha_1} := q^{-2}$.

PROPOSITION 4. *Let $\mathfrak{g} = A_2^{(2)}$. The $cr_{\Lambda_1}(m,n)$ satisfy the following recursion relations*:

S2.3. $cr_{\Lambda_1}(m, 2j) = cr_{\Lambda_1}(m-2, 2j) + (1+q^2)\, cr_{\Lambda_1}(m-2, 2j-1) + q^2 cr_{\Lambda_1}(m-2, 2j-2)$;

S2.4. $cr_{\Lambda_1}(m, 2j+1) = (q^{-1} + 1 + q)\, cr_{\Lambda_1}(m-2, 2j+1) + (q^{-2} + q^{-1} + 1 + q) cr_{\Lambda_1}(m-2, 2j) + q^{-2} cr_{\Lambda_1}(m-2, 2j-1)$.

Proof. We apply ar1–ar2 and br1–br6 to each summand of $cr_{\Lambda_1}(m,n) = \sum_{j \geq 0} a_r(m,j) b_r(j,n)$. ∎

6. DESCRIPTION OF PATHS FOR ALL OTHER MODULES

When our dominant weight λ is not one of the fundamental weights, we choose a dominant path that is a concatenation of L-S paths for the basic modules. (See [S] for discussion.) Let $\lambda = s\Lambda_0 + t\Lambda_1$. Set

$$\pi = \underbrace{\pi_{\Lambda_0} * \cdots * \pi_{\Lambda_0}}_{s} * \underbrace{\pi_{\Lambda_1} * \cdots * \pi_{\Lambda_1}}_{t} = \pi_{s\Lambda_0} * \pi_{t\Lambda_1}.$$

This path traces a straight path from $\pi(0)=0$ to $s\Lambda_0$ and then one from $s\Lambda_0$ to $\pi(1)=\lambda$. Note that this is an L-S path if and only if $st=0$. For two sets S_1, S_2 of paths, let $S_1 * S_2 := \{\pi_1 * \pi_2 | \pi_1 \in S_1,\ \pi_2 \in S_2\}$. For any two paths τ, σ note that

$$f_i(\tau * \sigma) = \begin{cases} \tau * (f_i\sigma) & \text{if } m_i(\tau) \geq \left\langle \tau(1), \alpha_i^\vee \right\rangle + m_i(\sigma); \\ (f_i\tau) * \sigma & \text{if } m_i(\tau) < \left\langle \tau(1), \alpha_i^\vee \right\rangle + m_i(\sigma). \end{cases}$$

From this, we see that there is a natural injection

$$B\pi \hookrightarrow \underbrace{B\pi_{\Lambda_0} * \cdots * B\pi_{\Lambda_0}}_{s} * \underbrace{B\pi_{\Lambda_1} * \cdots * B\pi_{\Lambda_1}}_{t}.$$

From [S], we have the following relations which describe exactly which paths lie in the image of $B\pi$. These relations will be crucial to the proof of Theorem 1.

(E1) $P_n(\pi_{k\Lambda}) = \bigcup_{0 \leq j \leq i \leq n} p_\Lambda(i,j) * P_j(\pi_{(k-1)\Lambda})$;

(E2) $P_{w_n^-}(\pi_{s\Lambda_0} * \pi_{t\Lambda_1}) = \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{i} p_{\Lambda_0}(i,j) * P_{w_{j+1}^-}(\pi_{(s-1)\Lambda_0} * \pi_{t\Lambda_1})$;

(E3) $P_{w_n^+}(\pi_{t\Lambda_1} * \pi_{s\Lambda_0}) = \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{i} p_{\Lambda_1}(i,j) * P_{w_{j+1}^+}(\pi_{(t-1)\Lambda_1} * \pi_{s\Lambda_0})$.

7. PROOF OF THEOREM 1

Proof for $\mathfrak{g} = A_1^{(1)}$. Recall the definition of real characters given in the Introduction. We obtain the real specialization by setting $e^{-\alpha_0} := q$ and $e^{-\alpha_1} := q^{-1}$ in R1.1–R1.4. Let $cr_\Lambda(m,n)$ be the specialized character of $p_\Lambda(m,n)$. This gives us the following recursion relations for $cr_{\Lambda_0}(m,n)$ and $cr_{\Lambda_1}(m,n)$:

S1.1. $cr_{\Lambda_0}(m,2j) = cr_{\Lambda_0}(m-2,2j) + (1+q^{-1})cr_{\Lambda_0}(m-2,2j-1) + q^{-1}cr_{\Lambda_0}(m-2,2j-2)$;

S1.2. $cr_{\Lambda_0}(m,2j+1) = cr_{\Lambda_0}(m-2,2j+1) + (1+q)cr_{\Lambda_0}(m-2,2j) + qcr_{\Lambda_0}(m-2,2j-1)$;

S1.3. $cr_{\Lambda_1}(m,2j) = cr_{\Lambda_1}(m-2,2j) + (1+q)cr_{\Lambda_1}(m-2,2j-1) + qcr_{\Lambda_1}(m-2,2j-2)$;

S1.4. $cr_{\Lambda_1}(m,2j+1) = cr_{\Lambda_1}(m-2,2j+1) + (1+q^{-1})cr_{\Lambda_1}(m-2,2j) + q^{-1}cr_{\Lambda_1}(m-2,2j-1)$.

We now establish the character formulas. By using the definition of L-S paths and relations E1–E3, we compute directly $\chi_r(E_w(\lambda))$ for $l(w) \leq 2$. Now consider the case when λ is singular. Without loss of generality, we can assume that $\lambda = s\Lambda_0$. Set $R_n := P_n(\pi_{s\Lambda_0}) \setminus P_{n-1}(\pi_{s\Lambda_0})$. By relation E1, we have that $\chi_r(R_n) = \sum_{j\geq 0} cr_{\Lambda_0}(n,j)\cdot\chi_r(P_j(\pi_{\Lambda_0}))$. Applying S1.1–S1.2 to each $cr_{\Lambda_0}(n,j)$ in this sum, we obtain $\chi_r(R_n) = q^{-s}[s+1]^2\chi_r(R_{n-2})$ for $n>2$. This proves our character formulas for $\chi_r(E_w(s\Lambda_0))$. An automorphism that exchanges α_0 and α_1 allows one to obtain $\chi_r(E_{w_n^-}(t\Lambda_1))$ from $\chi_r(E_{w_n^+}(s\Lambda_0))$.

Now let $\lambda = s\Lambda_0 + t\Lambda_1$ be regular. The proof is by induction on s and t. We first prove $\chi_r(E_w(\lambda))$ in the case when $w = w_n^-$. Set $R_{w_n^-} := P_{w_n^-}(\pi_{s\Lambda_0} * \pi_{t\Lambda_1}) \setminus P_{w_{n-1}^-}(\pi_{s\Lambda_0} * \pi_{t\Lambda_1})$. By relation E2, we have that $\chi_r(R_{w_n^-}) = \sum_{j\geq 0} cr_{\Lambda_0}(n-1,j) * \chi_r(P_{w_{j+1}^-}(\pi_{(s-1)\Lambda_0} * \pi_{t\Lambda_1}))$. By applying S1.1–S1.2 to each $cr_{\Lambda_0}(n-1,j)$ in the sum, we show that $\chi_r(R_{w_n^-}) = q^{-(s+t)}[s+t+1]^2\chi_r(R_{w_{n-2}^-})$. This proves the character formula for all λ and all w_n^-. Using the automorphism we obtain $\chi_r(E_w(\lambda))$ for $w = w_n^+$. ∎

Proof for $\mathfrak{g} = A_2^{(2)}$. We obtain the real specialization by setting $e^{-\alpha_0} := q$ and $e^{-\alpha_1} := q^{-2}$ in R2.1–R2.2. Let $cr_\Lambda(m,n)$ be the specialized character of $p_\Lambda(m,n)$. We have the recursion relations S2.3–S2.4 for $cr_{\Lambda_1}(m,n)$ and the following for $cr_{\Lambda_0}(m,n)$:

S2.1. $cr_{\Lambda_0}(m,2j) = cr_{\Lambda_0}(m-2,2j) + (1+q^{-1})\ cr_{\Lambda_0}(m-2,2j-1) + q^{-1}cr_{\Lambda_0}(m-2,2j-2)$;

S2.2. $cr_{\Lambda_0}(m,2j+1) = q(cr_{\Lambda_0}(m-2,2j) + cr_{\Lambda_0}(m-2,2j-1))$.

The rest of the proof uses the same methods as in the proof for $\mathfrak{g} = A_1^{(1)}$. In this case, however, one must consider each of the four cases ($\lambda = s\Lambda_0, t\Lambda_1, \lambda$ regular with $w = w_n^-, w = w_n^+$) separately as there is no automorphism that exchanges α_0 and α_1. ∎

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